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Ordinary smooth filters

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Abstract. First, we define an ordinary smooth filter and obtain its some properties. In particular, we obtain the necessary and sufficient condition of an ordinary smooth filter (See Theorem 3.9). Second, we define an ordinary smooth filter base (See Theorem 4.1). Third, we introduce the concept of an ordinary smooth ultrafilter and study its some properties. Moreover, we obtain the necessary and sufficient condition of an induced ordinary smooth filter (See Theorem 5.8). Fourth, we define the image and the inverse image of an ordinary smooth filter. Finally, we introduce level set and strong level set of an ordinary smooth filter and obtain their some properties.

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1. Introduction

In 1968, Chang [3] introduced the concept of fuzzy topology on a set X by using fuzzy sets introduced by Zadeh [19]. After then, Lowen [14], and Pu and Liu [16] updated it.

In 1985, Sostak [18] defined a fuzzy topology τ on a set X as a mapping $\tau : I^X \to I$ satisfying some axioms, where I^X denotes the set of all fuzzy sets in X. He considered the degree of openness of fuzzy sets, gave some basic rules and proved how such an extension can be done. In 1992, K. C. Chattopadhyay et al. [4] studied fuzzy topological spaces in Sostak's sense. In the same year, Ramadan [17] introduced similar concepts under the name of smooth topological spaces working in terms of lattices L and $L^{'}$ instead of $I = [0, 1]$. After that time, many researchers [5, 7, 8, 15, 20] investigated various properties of smooth topological spaces. In particular, Ying [20] studied fuzzifying topological spaces (called ordinary smooth topological spaces by Hur et al. [9]) considering of degree of openness of ordinary subsets. Furthermore, Hur et al. [10, 11, 12, 13] investigated various properties in ordinary smooth topological spaces, and Chae et al. [6] constructed the set $OST(X)$ of all ordinary smooth topologies on a set X and studied it in the sense of a lattice.

In this paper, first we define an ordinary smooth filter and obtain its some properties. In particular, we obtain the necessary and sufficient condition of an ordinary smooth filter (See Theorem 3.9). Second, we define an ordinary smooth filter base (See Theorem 4.1). Third, we introduce the concept of an ordinary smooth ultrafilter and study its some properties. Moreover, we obtain the necessary and sufficient condition of an induced ordinary smooth filter (See Theorem 5.8). Fourth, we define the image and the inverse image of an ordinary smooth filter. Finally, we introduce level set and strong level set of an ordinary smooth filter and obtain their some properties.

Throughout this paper, let $I = [0, 1]$ be the unit closed interval, and we will write $I_0 = (0.1]$ and $I_1 = [0.1)$.

2. Preliminaries

Let $2 = \{0, 1\}$ and let 2^X [resp. I^X] denote the set of all ordinary subsets [resp. fuzzy sets] of a set X.

Definition 2.1 ([9])**.** Let X be a non-empty set. Then a mapping $\tau : 2^X \to I$ is called an ordinary smooth topology (in short, *ost*) on X, if it satisfies the following axioms:

(OST_1) $\tau(\phi) = \tau(X) = 1$,
(OST_2) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$, for any $A,\ B \in 2^X$,
(OST_3) $\tau(\bigcup_{j\in J} A_j) \geq \bigwedge_{j\in J} \tau(A_j)$, for each $(A_j)_{j\in J} \subset 2^X$.

The pair (X, τ) is called an ordinary smooth topological space (in short, *osts*). We will denote the set of all $ost's$ on X as $OST(X)$.

Remark 2.2. Ying [20] called the mapping $\tau : 2^X \to I$ [resp. $\tau : I^X \to 2$ and $\tau : I^X \to I$] satisfying the axioms in Definition 2.1 as a fuzzifying topology [resp. fuzzy topology and bifuzzy topology] on X. In fact, the mapping $\tau : 2^X \to 2$ satisfying the axioms in Definition 2.1 is a classical topology on X.

Definition 2.3 ([20])**.** Let (X, τ) be an *osts* and let $x \in X$. Then a mapping $N_x : 2^X \to I$ is called an ordinary smooth neighborhood system of p w.r.t. τ, if for each $A \in 2^X$, $N_x = \bigvee_{x\in B\subset A} \tau(B)$.

3. Basic properties of ordinary smooth filters

Definition 3.1 ([1])**.** Let X be a set and let $\Im \subset 2^X$. Then $\Im$ is called a filter on X, if it satisfies the following axioms: for any $A,\ B \in 2^X$,

(F_I) if $A \in \Im$ and $A \subset B$, then $B \in \Im$,
(F_{II}) if $A,\ B \in \Im$, then $A \cap B \in \Im$,
(F_{III}) $X \in \Im$,
(F_{IV}) $\phi \notin \Im$.

We will denote the set of all filters on X as $F(X)$.

It is obvious that $\{X\} \in F(X)$.

Definition 3.2. Let X be a set. Then a mapping $\Im : 2^X \to I$ is called an ordinary smooth filter (in short, osf) on X, if it satisfies the following axioms: for any $A,\ B \in 2^X$,
(OSF_1) if $\phi \neq A \subset B$, then $\Im(A) \leq \Im(B)$,
(OSF_2) if $A \cap B \neq \phi$, then $\Im(A) \wedge \Im(B) \leq \Im(A \cap B)$,
(OSF_3) $\Im(X) = 1$,
(OSF_4) $\Im(\phi) = 0$.
The pair $(X, \Im)$ is called an ordinary smooth set filtered by $\Im$. We will denote the set of all ordinary smooth filters as $OSF(X)$.

From the conditions (OSF_1) and (OSF_2), it is obvious that for any $A,\ B \in 2^X$, if $A \cap B \neq \phi$ and $\phi \neq A \subset B$, then $\Im(A \cap B) = \Im(A) \wedge \Im(B)$.

Remark 3.3. (1) From Definition 3.2, it is clear that that $\Im$ is a fuzzy set in 2^X.

(2) Let $\Im \in OF(X)$. Then we can consider $\Im$ as the special mapping $\Im : 2^X \to I = \{0, 1\}$ satisfying all the axioms of Definition 3.1. Thus every filter on X is an ordinary smooth filter on X, i.e., $F(X) \subset OSF(X)$.

Example 3.4. (1) Let $X = \{a, b, c\}$. Then $2^X = \{\phi, \{a\}, \{b\}, \{c, \}, \{a, b\}\{b, c\}, \{b, c\}, X\}$. We define the mapping $\Im : 2^X \to I$ defined as follows:

$$\Im(\{a\}) = 0.7,\ \Im(\{b\}) = 0.7,\ \Im(\{c\}) = 0.8,\ \Im(\{a, b\}) = 0.7,$$
$$\Im(\{a, c\}) = 0.9,\ \Im(\{b, c\}) = 0.8,\ \Im(X) = 1,\ \Im(\phi) = 0.$$

Then we can easily see that $\Im \in OSF(X)$.

(2) Let X be a non-empty setand let $\phi \neq A \in 2^X$. We define two mappings $\Im_X,\ \Im_A : 2^X \to I$ as follows: for each $B \in 2^X$,

$$\Im_X(B) = \begin{cases} 1 & \text{if } B = X, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Im_A(B) = \begin{cases} 1 & \text{if } A \subset B, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can easily see that $\Im_X,\ \Im_A \in OSF(X)$.

(3) Let X be an infinite set. We define the mapping $\Im_f : 2^X \to I$ as follows: for each $A \in 2^X$,

$$\Im_f(A^c) = \begin{cases} 1 & \text{if } A \text{ is a non-empty finite subset of } X, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can easily prove that $\Im_f \in OSF(X)$. In this case, $\Im_f$ is called the ordinary smooth filter of the complements of the finite subsets of X. In particular, the ordinary smooth filter of the complements of the finite subsets of $\mathbb{N}$ is called the r-ordinary smooth Frechet filter and will be denoted by $\Im_{f,\mathbb{N}}$, where $\mathbb{N}$ denotes the set of all non-negative integers.

Definition 3.5. Let X be a non-empty set and let $\Im_1,\ \Im_2 \in OSF(X)$.

(i) $\Im_1$ is said to be finer than $\Im_2$ or $\Im_2$ is said to be coarser than $\Im_1$, denoted by $\Im_2 \leq \Im_1$, if $\Im_2(A) \leq \Im_1(A)$, $\forall A \in 2^X$.

(ii) $\Im_1$ is said to be strictly finer than $\Im_2$ or $\Im_2$ is said to be strictly coarser than $\Im_1$, denoted by $\Im_2 < \Im_1$, if $\Im_2(A) \leq \Im_1(A)$ and $\Im_2(A) \neq \Im_1(A)$, $\forall A \in 2^X$.

(iii) $\Im_1$ and $\Im_2$ are said to be comparable, if either $\Im_1(A) \leq \Im_2(A)$ or $\Im_2(A) \leq \Im_1(A)$.

It is obvious that $(OSF(X), \wedge)$ is a poset with the least element $\Im_X$.

Proposition 3.6. *Let X be a non-empty set and let $(\Im_j)_{j\in J}$ be a non-empty family of ordinary smooth filters on X. Then $\bigcap_{j\in J}\Im_j \in OSF(X)$, where $(\bigcap_{j\in J}\Im_j)(A) = \bigwedge_{j\in J}\Im_j(A)$.*

Proof. (OSF_1): For any $A,\ B \in 2^X - \{X, \phi\}$, suppose $A \subset B$. Then by the hypothesis, $\Im_j(A) \leq \Im_j(B)$, for each $j \in J$. Thus

$$(\bigcap_{j\in J}\Im_j)(A) = \bigwedge_{j\in J}\Im_j(A) \leq \bigwedge_{j\in J}\Im_j(B) = (\bigcap_{j\in J}\Im_j)(B).$$

So the axiom holds.

(OSF_2): Let $A,\ B \in 2^X - \{X, \phi\}$. Then by the hypothesis,

$$\Im_j(A) \wedge \Im_j(B) \leq \Im_j(A \cap B), \text{ for each } j \in J.$$

Thus $(\bigcap_{j\in J}\Im_j)(A) \wedge (\bigcap_{j\in J}\Im_j)(B) \leq \bigcap_{j\in J}(A\cap B)$. So the axiom holds.

(OSF_3), (OSF_4): The proofs are clear. □

Lemma 3.7. *Let $\Im \in OSF(X)$ and let $\mathfrak{S} : 2^X \to I$ be a mapping such that $\mathfrak{S} \leq \Im$. We define the mapping $\mathfrak{S}^{\sqcap} : 2^X \to I$ as follows: for each $A \in 2^X$,*

$$\mathfrak{S}^{\sqcap}(A) = \bigvee_{(S_j)_{j\in J}\sqsubset 2^X,\ A=\bigcap_{j\in J}S_j}\ \bigwedge_{j\in J}\mathfrak{S}(S_j),$$

where $\sqsubset$ stands for "a finite subset of". Then $\mathfrak{S}^{\sqcap}(\phi) = 0$.

Proof. Let $A \in 2^X$. Then

$$\begin{aligned}\mathfrak{S}^{\sqcap}(A) &= \textstyle\bigvee_{(S_j)_{j\in J}\sqsubset 2^X,\ A=\bigcap_{j\in J}S_j}\bigwedge_{j\in J}\mathfrak{S}(S_j)\\ &\leq \textstyle\bigvee_{(S_j)_{j\in J}\sqsubset 2^X,\ A=\bigcap_{j\in J}S_j}\bigwedge_{j\in J}\Im(S_j) \text{ [Since } \mathfrak{S}\leq\Im]\\ &\leq \textstyle\bigvee_{(S_j)_{j\in J}\sqsubset 2^X,\ A=\bigcap_{j\in J}S_j}\Im(\bigcap_{j\in J}(S_j))\\ &\quad \text{[By the axiom } (OSF_2) \text{ and induction]}\\ &\leq \Im(A).\end{aligned}$$

Since $\Im \in OSF(X)$, by the axiom (OSF_4), $\Im(\phi) = 0$. Thus $\mathfrak{S}^{\sqcap}(\phi) \leq \Im(\phi) = 0$. So $\mathfrak{S}^{\sqcap}(\phi) = 0$. □

Lemma 3.8. *Let $\mathfrak{S} : 2^X \to I$ be a mapping such that $\mathfrak{S}(X) = 1$ and let $\mathfrak{S}^{\sqcap} : 2^X \to I$ be the mapping given in Lemma 3.7. We define the mapping $\Im : 2^X \to I$ as follows: for each $A \in 2^X$,*

$$\Im(A) = \bigvee_{B\subset A}\mathfrak{S}^{\sqcap}(A).$$

If $\mathfrak{S}^{\sqcap}(\phi) = 0$, then $\Im \in OSF(X)$. Furthermore, $\Im$ is the coarsest ordinary smooth filter on X such that $\mathfrak{S} \leq \Im$.

Proof. (OSF_1): For any $A,\ B \in 2^X$, suppose $\phi \neq A \subset B$. Then

$$\begin{aligned}\Im(A) &= \textstyle\bigvee_{C\subset A}\mathfrak{S}^{\sqcap}(C) \text{ [By the definition of } \Im]\\ &\leq \textstyle\bigvee_{C\subset B}\mathfrak{S}^{\sqcap}(C) \text{ [Since } A\subset B]\\ &= \Im(B). \text{ [By the definition of } \Im]\end{aligned}$$

(OSF_2): Let $A,\ B \in 2^X$ such that $A \cap B \neq \phi$. Consider the sets

$$\mathfrak{C}_1 = \{C \in 2^X : C \subset A \cap B\} \text{ and } \mathfrak{C}_2 = \{C \in 2^X : C \subset A \text{ and } C \subset B\}.$$

Then we can easily see that $\mathfrak{C}_2 \subset \mathfrak{C}_1$. Thus by the definition of $\Im$,

$\Im(A \cap B) = \bigvee_{C \subset A \cap B} \mathfrak{S}^{\sqcap}(C)$
$= \bigvee_{C \in \mathfrak{C}_1} \mathfrak{S}^{\sqcap}(C)$
$\geq \bigvee_{C \in \mathfrak{C}_2} \mathfrak{S}^{\sqcap}(C)$ [Since $\mathfrak{C}_2 \subset \mathfrak{C}_1$]
$= (\bigvee_{C \subset A} \mathfrak{S}^{\sqcap}(C)) \wedge (\bigvee_{C \subset B} \mathfrak{S}^{\sqcap}(C))$
$= \Im(A) \wedge \Im(B)$.

(OSF_3): $\Im(X) = \mathfrak{S}^{\sqcap}(X)$
$\geq \mathfrak{S}^{\sqcap}(X)$
$= \bigvee_{(S_j)_{j \in J} \sqsubset 2^X, \ X = \bigcap_{j \in J} S_j} \bigwedge_{j \in J} \mathfrak{S}(S_j)$ [By the definition of $\mathfrak{S}^{\sqcap}$]
$\geq \mathfrak{S}(X)$ [If $J = \phi$, then $\bigcap_{j \in J} S_j = X$]
$= 1$. [By the hypothesis]

(OSF_4): It is clear that $\Im(\phi) = 0$. So $\Im \in OSF(X)$.

Now suppose $\Im^{'} \in OSF(X)$ such that $\mathfrak{S} \leq \Im^{'}$ and let $A \in 2^X$. Then

$\Im(A) = \mathfrak{S}^{\sqcap}(A)$
$= \bigvee_{(S_j)_{j \in J} \sqsubset 2^X, \ A = \bigcap_{j \in J} S_j} \bigwedge_{j \in J} \mathfrak{S}(S_j)$
$\leq \bigvee_{(S_j)_{j \in J} \sqsubset 2^X, \ A = \bigcap_{j \in J} S_j} \bigwedge_{j \in J} \Im^{'}(S_j)$
$\leq \Im^{'}(\bigcap_{j \in J} S_j)$ [Since $\Im^{'} \in OSF(X)$]
$= \Im^{'}(A)$. [Since $\exists (S_j)_{j \in J} \sqsubset 2^X$ such that $A = \bigcap_{j \in J} S_j$]
$\leq \Im^{'}(A)$. [Since $\Im^{'} \in OSF(X)$]

Thus $\Im$ is the coarsest ordinary smooth filter on X such that $\mathfrak{S} \leq \Im$. This completes the proof. □

The following is the immediate result of Lemmas 3.7 and 3.8.

Theorem 3.9. *Let $\mathfrak{S} : 2^X \to I$ be a mapping such that $\mathfrak{S}(X) = 1$. Then there is an $\Im \in OSF(X)$ if and only if $\mathfrak{S}^{\sqcap}(\phi) = 0$.*

In this case, $\Im$ is said to be generated by $\mathfrak{S}$ and $\mathfrak{S}$ is called an ordinary smooth subbase (in short, *ossb*) for $\Im$.

Example 3.10. Let (X, τ) be an *osts* and let N_x be an ordinary smooth neighborhood system of of $x \in X$ w.r.t. τ. Then clearly $N_x(X) = 1$. Assume that $N_x(\phi) \neq 0$, i.e., $N_x(\phi) > 0$. Then by Result 2.6,

$$N_x(\phi) = \bigvee_{x \in A \subset \phi} \tau(A) > 0.$$

Thus there is $A_0 \in 2^X$ such that $x \in A_0 \subset \phi$. This is a contradiction. So $N_x(\phi) = 0$. On the other hand,

$N_x^{\sqcap}(\phi) = \bigvee_{(S_j)_{j \in J} \sqsubset 2^X, \ \phi = \bigcap_{j \in J} S_j} \bigwedge_{j \in J} N_x(S_j)$
$\leq N_x(\phi)$
$= 0$.

Hence by Theorem 3.9, there is an $\Im_x \in OSF(X)$ such that N_x is an ordinary smooth subbase for $\Im_x$.

In this case, $\Im_x$ is called the ordinary smooth neighborhood filter of x for τ.

The following is the immediate result of Definition 3.2 and Theorem 3.9.

Corollary 3.11. *Let $\Im \in OSF(X)$ and let $A \in 2^X$. Then there is an $\Im^{'} \in OSF(X)$ such that $\Im \leq \Im^{'}$ and $\Im^{'}(A) \neq 0$ if and only if $A \cap B \neq \phi$, for each $B \in 2^X$ with $\Im(B) \neq 0$.*

Corollary 3.12. *Let X be a non-empty set and let $\Phi \subset OSF(X)$. Then there exists $\bigvee \Phi$ if and only if for each $(\Im_i)_{i \leq i \leq n} \subset \Phi$ and each $A_i \in 2^X$ with $\Im_i(A_i) \neq 0$ $(1 \leq i \leq n)$, $\bigcap_{i=1}^n A_i \neq \phi$, where $\bigvee \Phi$ denotes the least upper bound for Φ in $OSF(X)$.*

Proof. Let $\mathfrak{S} = \bigcup_{\Im \in \Phi} \Im$, where $\mathfrak{S} : 2^X \to I$ is the mapping defined by: for each $A \in 2^X$,

$$\mathfrak{S}(A) = \bigvee_{\Im \in \Phi} \Im(A).$$

Then clearly, $\mathfrak{S}(X) = 1$. On the other hand,

$$\begin{aligned}\mathfrak{S}^{\sqcap}(\phi) &= \textstyle\bigvee_{(S_j)_{j\in J} \sqsubset 2^X,\ \phi = \bigcap_{j\in J} S_j} \bigwedge_{j\in J} \mathfrak{S}(S_j) \\ &= \textstyle\bigvee_{(S_j)_{j\in J} \sqsubset 2^X,\ \phi = \bigcap_{j\in J} S_j} \bigwedge_{j\in J} \bigvee_{\Im \in \Phi} \Im(S_j) \\ &= \textstyle\bigvee_{\Im \in \Phi} \bigvee_{(S_j)_{j\in J} \sqsubset 2^X,\ \phi = \bigcap_{j\in J} S_j} \bigwedge_{j\in J} \Im(S_j) \\ &\leq \textstyle\bigvee_{\Im \in \Phi} \Im(\phi) \\ &= 0. \text{ [By the axiom } (OSF_4)]\end{aligned}$$

Thus by Theorem 3.9, there is an $\Im^{'} \in OSF(X)$ such that $\mathfrak{S}$ is an ordinary smooth subbase for $\Im^{'}$.

It is obvious that $\Im \leq \Im^{'}$, for each $\Im \in \Phi$. So $\Im^{'}$ is the least upper bound for Φ. □

The following is the immediate result of Corollary 3.12.

Corollary 3.13. *The ordered set of all ordinary smooth filters on a non-empty set X is inductive.*

4. Bases of an ordinary smooth filter

Theorem 4.1. *Let X be a set and let $\mathfrak{B} : 2^X \to I$ be any mapping. We define the mapping $\Im : 2^X \to I$ as follows: for each $A \in 2^X$,*

$$\Im(A) = \bigvee_{B \subset A} \mathfrak{B}(B).$$

Then $\Im \in OSF(X)$ if and only if $\mathfrak{B}$ satisfies the following conditions:

(OSB_1) $\mathfrak{B}(B_1) \wedge \mathfrak{B}(B_2) \leq \bigvee_{B \subset B_1 \cap B_2} \mathfrak{B}(B)$, for any $B_1,\ B_2 \in 2^X$,

(OSB_2) $\mathfrak{B}$ is a normal fuzzy set in 2^X, i.e., there is $A_0 \in 2^X$ such that $\mathfrak{B}(A_0) = 1$ and $\mathfrak{B}(\phi) = 0$.

In this case, $\mathfrak{B}$ is called an ordinary smooth filter base (in short, $osfb$) for $\Im$ and $\Im$ is said to be generated by $\mathfrak{B}$.

Proof. Suppose $\Im \in OSF(X)$.

(OSB_1): Let $B_1,\ B_2 \in 2^X$. Then

$$\begin{aligned}\textstyle\bigvee_{B \subset B_1 \cap B_2} \mathfrak{B}(B) &= \Im(B_1 \cap B_2) \text{ [By the definition of } \Im] \\ &\geq \Im(B_1) \wedge \Im(B_2) \text{ [By the hypothesis]} \\ &= (\textstyle\bigvee_{C_1 \subset B_1} \mathfrak{B}(C_1)) \wedge (\bigvee_{C_2 \subset B_2} \mathfrak{B}(C_2))\end{aligned}$$

$\geq \mathfrak{B}(B_1) \wedge \mathfrak{B}(B_2)$.

(OSB_2): Since $\Im \in OSF(X)$, it is clear that $1 = \Im(X) = \bigvee_{A \subset X} \mathfrak{B}(A)$. Then there is $A_0 \in 2^X$ such that $\mathfrak{B}(A_0) = 1$. Thus $\mathfrak{B}$ is a normal fuzzy set in 2^X.

On the other hand, $0 = \Im(\phi) = \bigvee_{A \subset \phi} \mathfrak{B}(A) \geq \mathfrak{B}(\phi)$. So $\mathfrak{B}(\phi) = 0$.

Conversely, suppose the necessary conditions hold.

(OSF_1): For any $A,\ B \in 2^X$, let $A \subset B$. Then

$$\Im(A) = \bigvee_{C \subset A} \mathfrak{B}(C) \leq \bigvee_{C \subset B} \mathfrak{B}(C) = \Im(B).$$

(OSF_2): Let $A,\ B \in 2^X$. Then

$$\begin{aligned}\Im(A) \wedge \Im(B) &= (\bigvee_{C \subset A} \mathfrak{B}(C)) \wedge \bigvee_{D \subset B} \mathfrak{B}(D)) \\ &= \bigvee_{C \subset A} \bigvee_{D \subset B} [\mathfrak{B}(C) \wedge \mathfrak{B}(D)] \\ &\leq \bigvee_{C \subset A} \bigvee_{D \subset B} \bigvee_{E \subset C \cap D} \mathfrak{B}(E) \text{ [By the condition } (OSB_1)] \\ &\leq \bigvee_{E \subset A \cap B} \mathfrak{B}(E) \\ &= \Im(A \cap B).\end{aligned}$$

(OSF_3): By (OSB_2), $\mathfrak{B}$ is a normal fuzzy set in 2^X. Then there is $A_0 \in 2^X$ such that $\mathfrak{B}(A_0) = 1$. Thus $\Im(X) = \bigvee_{B \subset X} \mathfrak{B}(B) \geq \mathfrak{B}(A_0) = 1$. So $\Im(X) = 1$.

(OSF_4): Since $\mathfrak{B}(\phi) = 0$, it is obvious that $\Im(\phi) = 0$.

This completes the proof. □

Definition 4.2. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two ordinary smooth filter bases on a set X. Then $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are said to be equivalent, if they generate the same osf.

From Theorem 4.1, it is obvious that if $\mathfrak{S}$ is an $ossb$ for an osf $\Im$, then $\mathfrak{S}^{\sqcap}$ is an $osfb$ for $\Im$.

Theorem 4.3. *Let $\Im \in OSF(X)$ and let $\mathfrak{B} : 2^X \to I$ be a mapping such that $\mathfrak{B} \leq \Im$. Then $\mathfrak{B}$ is an $osfb$ for $\Im$ if and only if $\Im(A) = \bigvee_{B \subset A} \mathfrak{B}(B)$, for each $A \in 2^X$.*

Proof. Suppose $\mathfrak{B}$ is an $osfb$ for $\Im$. Then by Theorem 3.1, it is clear.

Conversely, suppose the necessary condition holds and let $B_1,\ B_2 \in 2^X$. Then

$$\begin{aligned}\mathfrak{B}(B_1) \wedge \mathfrak{B}(B_2) &\leq \Im(B_1) \wedge \Im(B_2) \text{ [Since } \mathfrak{B} \leq \Im] \\ &\leq \Im(B_1 \cap B_2) \text{ [By the axiom } (OSF_2)] \\ &= \bigvee_{A \subset B_1 cap B_2} \mathfrak{B}(A). \text{ [By the hypothesis]}\end{aligned}$$

Thus $\mathfrak{B}$ satisfies the condition (OSB_1).

Since $\Im \in OSF(X)$, it is clear that $\Im(X) = 1$. Then by the hypothesis, $1 = \bigvee_{A \subset X} \mathfrak{B}(A)$. Thus there is $A_0 \in 2^X$ such that $\mathfrak{B}(A_0) = 1$. So $\mathfrak{B}$ is a normal fuzzy set in 2^X. On the other hand, it is obvious that $\mathfrak{B}(\phi) = 0$. Hence $\mathfrak{B}$ satisfies the condition (OSB_2). This completes the proof. □

The following is the immediate result of Definition 3.5 and Theorem 4.1.

Corollary 4.4. *Let $\Im,\ \Im^{'} \in OSF(X)$ and let $\mathfrak{B}$ [resp. $\mathfrak{B}^{'}$] be an $osfb$ for $\Im$ [resp. $\Im^{'}$]. Then $\Im^{'}$ is finer than $\Im$ if and only if $\mathfrak{B}(B) = \bigvee_{B^{'} \subset B} \mathfrak{B}^{'}(B^{'})$, for each $B \in 2^X$.*

The following is the immediate result of Definition 4.2, Theorem 4.1 and Corollary 4.4.

Corollary 4.5. *Let $\mathfrak{B}$ and $\mathfrak{B}^{'}$ be two ordinary smooth filter bases on a set X. Then $\mathfrak{B}$ and $\mathfrak{B}^{'}$ are equivalent if and only if the following conditions are satisfied:*
(1) $\mathfrak{B}(B) = \bigvee_{B^{'} \subset B} \mathfrak{B}^{'}(B^{'})$, *for each* $B \in 2^X$,
(2) $\mathfrak{B}^{'}(B^{'}) = \bigvee_{B \subset B^{'}} \mathfrak{B}(B)$, *for each* $B^{'} \in 2^X$.

5. Ordinary smooth ultrafilters

Definition 5.1. Let X be a set. Then a mapping $\mathfrak{U} : 2^X \to I$ is called an ordinary smooth ultrafilter (in shot, $osuf$), if $\mathfrak{U}$ is a maximal element of $(OSF(X), \leq)$.

We will denote the set of all $osuf's$ on X as $OSUF(X)$.

It is well-known (See Zorn's Lemma in [2]) that every inductive ordered set has at least one maximal element.

The following is the immediate result of Corollary 3.13.

Proposition 5.2. *Let X be a set. If $\Im \in OSF(X)$, then there is an $osuf$ $\mathfrak{U}$ such that $\Im \leq \mathfrak{U}$.*

Proposition 5.3. *Let X be a set, let $\mathfrak{U} \in OSUF(X)$ and let $A,\ B \in 2^X$. If $\mathfrak{U}(A \cup B) \neq 0$, then either $\mathfrak{U}(A) \neq 0$ or $\mathfrak{U}(B) \neq 0$.*

Proof. Assume that there are $A,\ B \in 2^X$ such that $\mathfrak{U}(A \cup B) \neq 0$, $\mathfrak{U}(A) = 0$ and $\mathfrak{U}(B) = 0$. We define the mapping $\Im : 2^X \to I$ as follows: for each $M \in 2^X$,

$$\Im(M) = \bigvee_{\mathfrak{U}(A \cup M) \neq 0} \mathfrak{U}(M).$$

Then we can easily see tat $\Im \in OSF(X)$. Moreover, it is clear that $\mathfrak{U} \lneqq \Im$. This is a contradiction from the fact that $\mathfrak{U} \in OSUF(X)$. Thus the result holds. □

The following is the immediate result of Proposition 5.3 and the induction.

Corollary 5.4. *Let $\mathfrak{U} \in OSUF(X)$ and let $(A_i)_{1 \leq i \leq n} \subset 2^X$. If $\mathfrak{U}(\bigcup_{i=1}^n A_i) \neq 0$, then there is $i \in \{1, \cdots, n\}$ such that $\mathfrak{U}(A_i) \neq 0$.*

Proposition 5.5. *Let $\mathfrak{S}$ be an $ossb$ for an osf on a set X If either $\mathfrak{S}(Y) \neq 0$ or $\mathfrak{S}(Y^c) \neq 0$, for each $Y \in 2^X$, then $\mathfrak{S} \in OSUF(X)$.*

Proof. Let $\Im \in OSF(X)$ such that $\mathfrak{S} \leq \Im$ and let $Y \in 2^X$. Suppose $\Im(Y) \neq 0$. Then by the axiom (OSB_2), $0 = \Im(\phi) = \Im(Y \cap Y^c) \geq \Im(Y) \wedge \Im(Y^c)$. Thus $\Im(Y^c) = 0$. Since $\mathfrak{S} \leq \Im$, $\mathfrak{S}(Y^c) = 0$. By the hypothesis, $\mathfrak{S}(Y) \neq 0$. So $\Im(Y) \leq \mathfrak{S}(Y)$, i.e., $\Im \leq \mathfrak{S}$. Hence $\Im = \mathfrak{S}$. Therefore $\mathfrak{S} \in OSUF(X)$. □

Example 5.6. Let X be a non-empty set and let $a \in X$ be fixed. We define the mapping $\Im : 2^X \to I$ as follows: for each $A \in 2^X$,

$$\Im(A) = \begin{cases} 1, & \text{if } a \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then we can easily show that $\Im \in OSF(X)$. Furthermore, it is clear that either $\Im(Y) \neq 0$ or $\Im(Y^c) \neq 0$, for each $Y \in 2^X$. Thus by Proposition 5.5, $\Im \in OSUF(X)$.

Proposition 5.7. *Let X be a set, let $\Im \in OSF(X)$ and let $(\mathfrak{U}_j)_{j \in J} \subset OSUF(X)$ such that $\Im \leq \mathfrak{U}_j$, for each $j \in J$. Then $\Im = \bigcap_{j \in J} \mathfrak{U}_j$, where $(\bigcap_{j \in J} \mathfrak{U}_j)(A) = \bigwedge_{j \in J} \mathfrak{U}_j(A)$, for each $A \in 2^X$.*

Proof. It is obvious that $\Im \leq \bigcap_{j\in J} \mathfrak{U}_j$. Let $A \in 2^X$ such that $\Im(A) = 0$. Assume that $B \subset A$, for each $B \in 2^X$ such that $\Im(B) \neq 0$. By the axiom (OSF_1), $\Im(B) \leq \Im(A) = 0$. Then $\Im(B) = 0$. This is a contradiction. Thus $B \not\subset A$, for each $B \in 2^X$ such that $\Im(B) \neq 0$, i.e., $B \cap A^c \neq \phi$, for each $B \in 2^X$ such that $\Im(B) \neq 0$. By Corollary 3.11, there is an $\Im^{'} \in OSF(X)$ such that $\Im \leq \Im^{'}$ and $\Im^{'}(A^c) \neq 0$. By Proposition 5.2, there is $\mathfrak{U} \in OSUF(X)$ such that $\Im^{'} \leq \mathfrak{U}$. So $\mathfrak{U}(A) = 0$, i.e., $(\bigcap_{j\in J} \mathfrak{U}_j)(A) = 0$. Hence $\bigcap_{j\in J} \mathfrak{U}_j \leq \Im$. Therefore $\Im = \bigcap_{j\in J} \mathfrak{U}_j$. □

Theorem 5.8. *Let X be a set, let $\Im \in OSF(X)$ and let $A \in 2^X$. We define the mapping $\Im_A : 2^X \to I$ as follows: for each $B \in 2^X$,*

$$\Im_A(B) = \begin{cases} \Im(A \cap B), & \text{if } A \not\subset B, \\ 1, & \text{if } A \subset B. \end{cases}$$

Then $\Im_A$ is an osf on A if and only if $A \cap M \neq \phi$, for each $M \in 2^X$ such that $\Im(M) \neq 0$.

In this case, $\Im_A$ is said to be induced by $\Im$ on A.

Proof. Suppose $\Im_A$ is an osf on A. Then the proof is clear.

Conversely, suppose the necessary condition holds. From the definition of $\Im_A$, it is obvious that $\Im_A(A) = 1$ and $\Im_A(\phi) = 0$. Then $\Im_A$ satisfies the axioms (OSF_3) and (OSF_4).

For any $B,\ C \in 2^X$, suppose $B \subset C$.

Case (i): If $A \subset B$, then clearly, $\Im_A(B) = 1 = \Im_A(C)$.

Case (ii): If $A \not\subset B$, then by the axiom (OSF_1),

$$\Im_A(B) = \Im(A \cap B) \leq \Im(A \cap C) = \Im_A(C).$$

Thus $\Im_A$ satisfies the axiom (OSF_1).

Now let $B,\ C \in 2^X$. Then

$$\begin{aligned} \Im_A(B \cap C) &= \Im(A \cap (B \cap C)) \text{ [By the definition of } \Im_A] \\ &= \Im((A \cap B) \cap (A \cap C)) \\ &\geq \Im(A \cap B) \wedge \Im(A \cap C) \text{ [By the axiom } (OSF_2)] \\ &= \Im_A(B) \wedge \Im_A(C). \end{aligned}$$

Thus $\Im_A$ satisfies the axiom (OSF_2).

This completes the proof. □

Remark 5.9. Let $\Im \in OSF(X)$, let $\mathfrak{B}$ be an $osfb$ for $\Im$ and let $A \in 2^X$. If $\Im$ induces an osf on A, then by Theorem 4.3, $\mathfrak{B}_A$ is an $osfb$ for $\Im_A$, where $\mathfrak{B}_A : 2^X \to I$ is the mapping defined by $\mathfrak{B}_A(B) = \mathfrak{B}(A \cap B)$, for each $B \in 2^X$.

The following is the immediate result of Propositions 5.3 and 5.5.

Theorem 5.10. *Let X be a set, let $\mathfrak{U} \in OSUF(X)$ and let $A \in 2^X$. Then $\mathfrak{U}$ induce an osf on A if and only if $\mathfrak{U}(A) \neq 0$. If this condition is satisfied, then $\mathfrak{U}_A \in OSUF(A)$, where $\mathfrak{U}_A : 2^X \to I$ is the mapping defined by $\mathfrak{U}_A(B) = \mathfrak{U}(A \cap B)$, for each $B \in 2^X$.*

6. The image and the inverse image of an ordinary smooth filter

Proposition 6.1. *Let X and Y be sets and let $f : X \to Y$ be a mapping. If $\mathfrak{B}$ is an $osfb$ on X, then $f(\mathfrak{B})$ is an $osfb$ on Y, where $f(\mathfrak{B}) : 2^Y \to I$ is the mapping defined as follows: for each $B \in 2^Y$,*

$$[f(\mathfrak{B})](B) = \bigvee_{A \subset f^{-1}(B)} \mathfrak{B}(A).$$

In this case, $f(\mathfrak{B})$ is called a the image of $\mathfrak{B}$ under f.

Proof. Suppose $\mathfrak{B}$ is an $osfb$ on X and let $B_1,\ B_2 \in 2^Y$.

Case (i): If either $f^{-1}(B_1) = \phi$ or $f^{-1}(B_2) = \phi$, then clearly,

$$[f(\mathfrak{B})](B_1) \wedge [f(\mathfrak{B})](B_2) = 0 \leq \bigvee_{B \subset B_1 \cap B_2} [f(\mathfrak{B})](B).$$

Case (ii): If either $f^{-1}(B_1) \neq \phi$ and $f^{-1}(B_2) \neq \phi$, then

$$\begin{aligned}
&[f(\mathfrak{B})](B_1) \wedge [f(\mathfrak{B})](B_2) \\
&= (\textstyle\bigvee_{A_1 \subset f^{-1}(B_1)} \mathfrak{B}(A_1)) \wedge (\bigvee_{A_2 \subset f^{-1}(B_2)} \mathfrak{B}(A_2)) \\
&= \textstyle\bigvee_{A_1 \subset f^{-1}(B_1),\ A_2 \subset f^{-1}(B_2)} [\mathfrak{B}(A_1) \wedge \mathfrak{B}(A_2)] \\
&\leq \textstyle\bigvee_{A_1 \subset f^{-1}(B_1),\ A_2 \subset f^{-1}(B_2)} \bigvee_{A \subset A_1 \cap A_2} \mathfrak{B}(A) \text{ [By the condition } (OSB_1)] \\
&\leq \textstyle\bigvee_{A_1 \cap A_2 \subset f^{-1}(B_1 \cap B_2)} \bigvee_{A \subset A_1 \cap A_2} \mathfrak{B}(A) \\
&\quad [\text{Since } f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)] \\
&\leq \textstyle\bigvee_{f(A) \subset B_1 \cap B_2} \bigvee_{A \subset f^{-1}(f(A))} \mathfrak{B}(A) \\
&= \textstyle\bigvee_{f(A) \subset B_1 \cap B_2} [f(\mathfrak{B})](f(A)).
\end{aligned}$$

Thus in either case, $f(\mathfrak{B})$ satisfies the condition (OSB_1).

From the definition of $f(\mathfrak{B})$, it is clear that $[f(\mathfrak{B})](\phi) = 0$. Since $\mathfrak{B}$ is a normal fuzzy set in 2^X, there is $A_0 \in 2^X$ such that $\mathfrak{B}(A_0) = 1$. Then $A_0 \neq \phi$. Thus there is $B \in 2^Y$ such that $A_0 \subset f^{-1}(B)$. So by the definition of $f(\mathfrak{B})$,

$$[f(\mathfrak{B})](B) = \bigvee_{A \subset f^{-1}(B)} \mathfrak{B}(A) \geq \mathfrak{B}(A_0) = 1.$$

Hence $f(\mathfrak{B})$ is a normal fuzzy set in 2^Y. Therefore $f(\mathfrak{B})$ satisfies the condition (OSB_2).

This is completes the proof. □

The following is the immediate result of Corollary 4.4 and Proposition 6.1.

Corollary 6.2. *Let X and Y be sets, let $f : X \to Y$ be a mapping and let $\mathfrak{B}$ be an $osfb$ on X. If $\mathfrak{B}^{'}$ is an $osfb$ for an osf which is finer than the osf for $\mathfrak{B}$, then $f(\mathfrak{B}^{'})$ is an $osfb$ for an osf finer than the osf for $f(\mathfrak{B})$.*

Theorem 6.3. *Let X and Y be sets, let $f : X \to Y$ be a mapping and let $\mathfrak{B}^{'}$ be an $osfb$ on Y. Then $f^{-1}(\mathfrak{B}^{'})$ is an $osfb$ on X if and only if $f^{-1}(M^{'}) \neq \phi$, i.e., $M^{'} \cap f(X) \neq \phi$, for each $M^{'} \in 2^Y$ such that $\mathfrak{B}^{'}(M^{'}) \neq 0$, where $f^{-1}(\mathfrak{B}^{'}) : 2^X \to I$ is the mapping defined by:*

$$[f^{-1}(\mathfrak{B}^{'})](A) = \mathfrak{B}^{'}(f(A)),\ for\ each\ A \in 2^X.$$

In this case, $f^{-1}(\mathfrak{B}^{'})$ is called a inverse image of $\mathfrak{B}^{'}$ under f.

Proof. This is an immediate consequence of the relation $f^{-1}(M^{'} \cap N^{'}) = f^{-1}(M^{'}) \cap f^{-1}(N^{'})$, for any $M^{'},\ N^{'} \in 2^Y$ and Theorem 4.1. □

Remark 6.4. (1) Let X and Y be sets, let $f : X \to Y$ be a mapping and let $\mathfrak{B}^{'}$ be an $osfb$ on Y. If $f^{-1}(M^{'}) \neq \phi$, for each $M^{'} \in 2^Y$ such that $\mathfrak{B}^{'}(M^{'}) \neq 0$, then from Theorem 6.4, $f(f^{-1}(M^{'}))$ is an $osfb$ for an osf finer than the osf for $\mathfrak{B}^{'}$.

(2) Let X and Y be sets and let $f : X \to Y$ be a mapping. If $\mathfrak{B}$ is an $osfb$ on X, then Proposition from 6.1 and Theorem 6.4, $f^{-1}(f(\mathfrak{B}))$ is an $osfb$ for an osf coarser than the osf for $\mathfrak{B}$.

7. r-level and strong r-level of an ordinary smooth filter

Definition 7.1. Let $\Im \in OSF(X)$ and let $r \in I$. Then r-level set and strong r-level set of $\Im$, denoted by $[\Im]_r$ and $[\Im]^*_r$, are sets of ordinary subsets of X defined as follows:

$$[\Im]_r = \{A \in 2^X : \Im(A) \geq r\}$$

and

$$[\Im]^*_r = \{A \in 2^X : \Im(A) > r\}.$$

The following is the similar result to Result 2.4.

Proposition 7.2. *Let $\Im \in OSF(X)$ and let $\Im(X)$ denote the set of all classical filters on X. Then*

(1) $[\Im]_r \in \Im(X)$, $\forall r \in I_0$,

$(1)^{'}$ $[\Im]^*_r \in \Im(X)$, $\forall r \in I_1$,

(2) *for any $r,\ s \in I$, if $r \leq s$, then $[\Im]_r \subset [\Im]_s$ and $[\Im]^*_r \subset [\Im]^*_s$.*

(3) $[\Im]_r = \bigcap_{s<r}[\Im]_s$, $\forall r \in I_0$,

$(3)^{'}$ $[\Im]^*_r = \bigcup_{s>r}[\Im]^*_s$, $\forall r \in I_1$

Proof. The proofs of (1), $(1)^{'}$ and (2) are obvious from Definitions 3.1 and 7.1.

(3) From (2), it is clear that $\{[\Im]_r : r \in I\}$ is a descending family of classical filters on X. Let $r \in I_0$. Then clearly, $[\Im]_r \subset \bigcap_{s<r}[\Im]_s$. Assume that $A \notin [\Im]_r$. Then $\Im(A) < r$. Thus $\exists\ s \in I_0$ such that $\Im(A) < s < r$. So $A \notin [\Im]_s$, for some $s < r$, i.e., $A \notin \bigcap_{s<r}[\Im]_s$. Hence $\bigcap_{s<r}[\Im]_s \subset [\Im]_r$. Therefore $[\Im]_r = \bigcap_{s<r}[\Im]_s$.

$(3)^{'}$ The proof is similar to (3). □

Proposition 7.3. *Let $\{\Im_r : r \in I\}$ be a non-empty descending family of classical filters on a set X.*

(1) *We define the mapping $\Im : 2^X \to I$ as follows: for each $A \in 2^X$,*

$$\Im(A) = \begin{cases} \bigvee r, & \text{if } A \in \Im_r, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\Im \in OSF(X)$.

(2) *For each $r \in I_0$, if $\Im_r = \bigcap_{s<r} \Im_s$, then $[\Im]_r = \Im_r$.*

$(2)^{'}$ *For each $r \in I_1$, if $\Im_r = \bigcup_{s>r} \Im_s$, then $[\Im]^*_r = \Im_r$.*

In this case, $\Im$ is called the ordinary smooth filter generated by $\{\Im_r : r \in I\}$.

Proof. (1) By Definition of $\Im$, it is clear that $\Im(X) = 1$ and $\Im(\phi) = 0$. Then $\Im$ satisfies the axioms (OSF_3) and (OSF_4). Also, by Definition of $\Im$, it is obvious that for any $A,\ B \in 2^X$, if $\phi \neq A \subset B$, then $\Im(A) \leq \Im(B)$. Thus $\Im$ satisfies the axiom (OSF_1).

For any $A_i \in 2^X$, let $\Im(A_i) = k_i,\ i = 1,\ 2$. Suppose $k_i = 0$, for some i. Then clearly, $\Im(A_1 \cap A_2) \geq \Im(A_1) \wedge \Im(A_2)$. Thus, without loss of generality, suppose $k_i > 0$, for $i = 1,\ 2$. Let $\epsilon > 0$. Then $\exists r_i \in I_0$ such that $k_i - \epsilon < r_i < k_i$ and $A_i \in \Im_{r_i},\ i = 1,\ 2$. Let $r = r_1 \wedge r_2$ and let $k = k_1 \wedge k_2$. Since $\{\Im_r : r \in I\}$ is a descending family, $A_i \in \Im_{r_i}$ and $A_1,\ A_1 \in \Im_r$. Thus $A_1 \cap A_2 \in \Im_r$. So by the definition of $\Im$, $\Im(A_1 \cap A_2) \geq r > k - \epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that

$$\Im(A_1 \cap A_2) \geq k = k_1 \wedge k_2 = \Im(A_1) \wedge \Im(A_2).$$

Hence $\Im$ satisfies the axiom (OSF_2).

(2) Suppose $\Im_r = \bigcap_{s<r} \Im_s$, for each $r \in I_0$ and let $A \in \Im_r$. Then clearly, $\Im(A) \geq r$. Thus $A \in [\Im]_r$. So $\Im_r \subset [\Im]_r$, for each $r \in I_0$.

Now let $A \in [\Im]_r$. Then $\Im(A) \geq r$. Thus by the definition of $\Im$,

$$\Im(A) = \bigvee_{A \in \Im_s} s \geq r.$$

Let $\epsilon > 0$. Then $\exists k \in I_0$ such that $s - \epsilon < k$ and $A \in \Im_k$. Thus

$$r - \epsilon \leq s - \epsilon < k \text{ and } A \in \Im_k.$$

So $A \in \Im_{r-\epsilon}$. Since $\epsilon > 0$ is arbitrary, $A \in \Im_r$. Hence $[\Im]_r \subset \Im_r$. Therefore $[\Im]_r = \Im_r$, for each $r \in I_0$.

$(2)'$ the proof is similar to (2). $\square$

The following is the immediate result of Propositions 7.2 and 7.3.

Corollary 7.4. *Let X be a non-empty set, let $\Im \in OSF(X)$ and let $\{[\Im]_r : r \in I\}$ be the family of all r-level classical filters w.r.p. $\Im$. Define the mapping $\Im_1 : 2^X \to I$ as follows: for each $A \in 2^X$,*

$$\Im_1(A) = \begin{cases} \bigvee r, & \text{if } A \in [\Im]_r, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\Im_1 = \Im$.

The fact that an ordinary smooth filter is fully determined by its decomposition in classical filters is restated in the following result.

Corollary 7.5. *Let X be a non-empty set and let $\Im_1,\ \Im_2 \in OSF(X)$. Then*

$\Im_1 = \Im_2$ if and only if $[\Im_1]_r = [\Im_2]_r$ or alternatively, $[\Im_1]^*_r = [\Im_2]^*_r$, for each $r \in I$.

Definition 7.6. Let X be a non-empty set, let $F \in F(X)$ and let $\Im \in OSF(X)$. Then $\Im$ is said to be compatible with F, if $F = S(\Im) = \{A \in 2^X : \Im(A) > 0\}$.

Example 7.7. (1) Let $\Im_X$ be the ordinary smooth filter on X given in Example 3.4 (2). Then clearly, $S(\Im_X) = \{X\}$. Thus $\Im_X$ is compatible with the classical filter $\{X\}$.

(2) Let $\Im_{f,\mathbb{N}}$ be the ordinary smooth Frechet filter given in Example 3.4 (3). Then we can easily see that $\Im_{f,\mathbb{N}}$ is compatible with the classical Frechet filter.

8. Conclusions

We introduced the concepts of an ordinary smooth filter, an ordinary smooth filter base, an ordinary smooth ultrafilter and an induced ordinary smooth filter and a (strong) r-level set, and studied some of its properties, respectively.

In the future, we will investigate the product of two ordinary smooth filter bases, a limit of an ordinary smooth filter and a limit of a mapping with respect to an ordinary smooth filter.

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